Numerical reconstruction of solutions in inverse problems for partial differential equations

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Reserch Interests

- **Numerical reconstruction of unknown boundaries based on the enclosre method (with Prof. M. Ikehata (Hiroshima Univ))**
    - **Electrical impedance tomography**
    - **Inverse scattering problem**
- **Numerical reconstruction of unknown sources based on the concept of reciprocity gap**
    - **Inverse source problem for the Poisson equation**
    - **Inverse source problem for wave equation**
- **I am interested in a "Direct" numerical method for inverse problems.**

"Direct" method vs "Indirect" method

- "Indirect" method
  - Estimate unknown parameters using an optimization procedure for some objective (or cost) function.
  - We need to solve the forward problem iteratively.
- "Direct" method
  - Estimate unknown parameters using a kind of "reconstruction formula".
  - We do NOT need to solve the forward problem.

Recent result: Reconstruction of moving point wave sources in a scalar wave equation

- Ω : **bounded domain in** $\mathbb{R}^3$
- $\Gamma = \partial\Omega$: **the boundary of** Ω **(**C^∞**-class)**
- $F(\mathbf{r}, t)$: **the source term (unknown)** $(\mathbf{r} = (x, y, z) \in \Omega,\ t \in (0, T))$
- $u(\mathbf{r}, t)$: **the solution of the initial and boundary value problem**

$$
(E) \begin{cases}
\dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2}(\mathbf{r}, t) - \Delta u(\mathbf{r}, t) = F(\mathbf{r}, t), & (\mathbf{r}, t) \in \Omega \times (0, T), \\
u(\mathbf{r}, t) = 0, & (\mathbf{r}, t) \in \Gamma \times (0, T), \\
u(\mathbf{r}, 0) = 0, & \mathbf{r} \in \Omega, \\
\dfrac{\partial u}{\partial t}(\mathbf{r}, 0) = 0, & \mathbf{r} \in \Omega,
\end{cases}
$$

($c > 0,\ \ T > 0$**: given constants)**

- **The source term : linear combination of moving point wave sources**

$$F(\mathbf{r},t)=\sum_{m=1}^{M}\lambda_m(t)\delta_{\mathbf{p}_m(t)}(\mathbf{r})$$

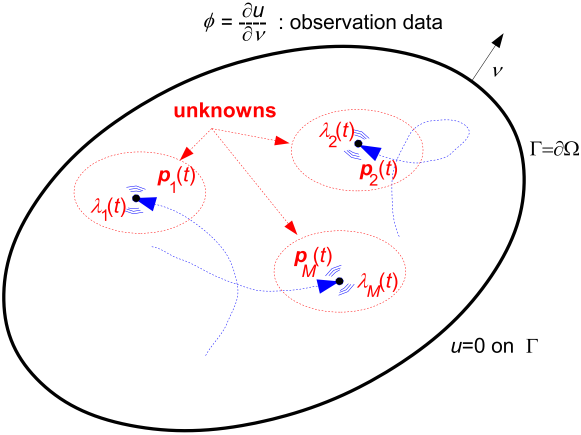
  - M **: number of point sources (unknown)**
  - $\mathbf{p}_m(t)=(p_{m,x}(t),p_{m,y}(t),p_{m,z}(t))\in C^2(0,T)$ **:**
    location of m**-th point source (unknown)**
    (Assume $|\dot{\mathbf{p}}_m(t)|<c$**)**
  - $\lambda_m(t)$ **: magnitude of** m**-th point source (unknown)**
    $\lambda_m\in C[0,T]\bigcap C^1(0,T)$, $\lambda_m(0)=0$

- **Observations**
  - **The normal derivative of** $u(\mathbf{r},t)$ **on** Γ**, i.e.**

$$\frac{\partial u}{\partial \nu}(\mathbf{r},t),\quad (\mathbf{r},t)\in\Gamma\times(0,T)$$

Our Goal

Estimate M, $\mathbf{p}_m(t)$, **and** $\lambda_m(t)$, $m=1,2,\cdots,M$



$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = F(\boldsymbol{x},t)$$

$$F(\boldsymbol{x},t)=\sum_{m=1}^{M} \lambda_m(t)\delta_{\boldsymbol{p}_m(t)}(\boldsymbol{x},t)$$

Reciprocity Gap Functional

- $\mathcal{R}(\cdot)$: **the reciprocity gap functional**

$$\mathcal{R}(v) = \frac{1}{c^2}\int_\Omega \frac{\partial u}{\partial t}(\mathbf{r},T)v(\mathbf{r},T)dV(\mathbf{r}) - \frac{1}{c^2}\int_\Omega u(\mathbf{r},T)\frac{\partial v}{\partial t}(\mathbf{r},T)dV(\mathbf{r}) - \int_0^T\int_\Gamma \frac{\partial u}{\partial \nu}(\mathbf{r},t)v(\mathbf{r},t)dS(\mathbf{r})dt,$$

  v: **a complex-valued function which satisfies the homogeneous wave equation**

$$\frac{1}{c^2}\frac{\partial^2 v}{\partial t^2}(\mathbf{r},t) - \Delta v(\mathbf{r},t) = 0, \quad (\mathbf{r},t)\in\Omega\times(0,T)$$

- **We can establish the following relation between the reciprocity gap functional** $\mathcal{R}(v)$ **and the source term** $F(\mathbf{r},t)$**:**

$$\mathcal{R}(v) = \int_0^T\int_\Omega F(\mathbf{r},t)v(\mathbf{r},t)dV(\mathbf{r})dt$$

- **Choose v as follwing:**

$$
\begin{aligned}
f_n(\mathbf{r},t;\tau,\varepsilon) &= (x+\mathrm{i}y)^n\rho_\varepsilon\left(t+\tfrac{z}{c}-\tau\right), \quad n=0,1,2\cdots \\
g_n(\mathbf{r},t;\tau,\varepsilon) &= -\frac{\partial}{\partial t}f_n(\mathbf{r},t;\tau,\varepsilon), \quad n=0,1,2,\cdots \\
h_n(\mathbf{r},t;\tau,\varepsilon) &= z\left(\frac{\partial}{\partial x}-\mathrm{i}\frac{\partial}{\partial y}\right)f_n(\mathbf{r},t;\tau,\varepsilon) \\
&\quad -(x-\mathrm{i}y)\frac{\partial}{\partial z}f_n(\mathbf{r},t;\tau,\varepsilon), \quad n=1,2,\cdots
\end{aligned}
$$

where ρ_ε: a mollifier function with support $[-\varepsilon,\varepsilon]$
(Note: This choice is the same with O-Inui-Ohnaka(2011).)

- **We can obtain algebraic reconstruction formula for unknown parameters $\mathbf{p}_m(t)$, and $\lambda_m(t)$ from reciprocity gap functionals for these functions.**

Numerical experiment

- $\Omega = \{\mathbf{r} \mid |\mathbf{r}| = 1\}$
- **time interval:** $0 \leq t \leq T = 40$
- $c = 1$ **: wave propagation speed**
- **Locations $\mathbf{p}_m(t)$, and magnitudes $\lambda_m(t)$:**

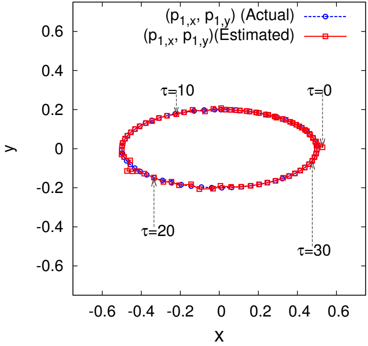
wave source 1 $\mathbf{p}(t) = (0.5\cos(0.2t),\ 0.2\sin(0.2t),\ 0.2\sin(0.45t))$

$$\lambda_1(t) = \frac{1}{2}\sin\frac{\pi}{9}t$$

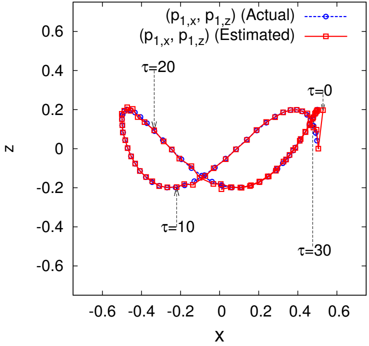
wave source 2 $\mathbf{p}(t) =$

$$(r(t)\cos\theta(t),\ r(t)\sin\theta(t)\cos(0.7\pi),\ r(t)\sin\theta(t)\sin(0.7\pi))$$
$$\theta(t) = (2\pi - 2\theta_0)t/50 + \theta_0,\ (\theta_0 = \cos^{-1} 0.6875)$$
$$r(t) = 0.25/(1.0 - \cos(\theta(t)))$$
$$\lambda_2(t) = \begin{cases} 1.0 - \cos(2\pi(t - 5.0)/19.5)) & 5.0 \leq t \leq 24.5 \\ 0 & \textbf{others} \end{cases}$$

Estimation result for location of wave source 1

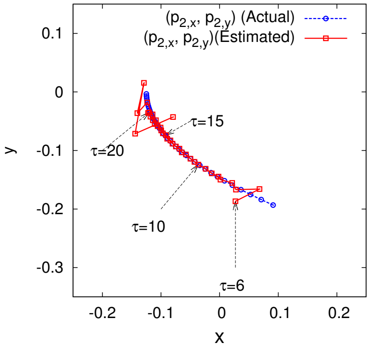


xy-**plane**

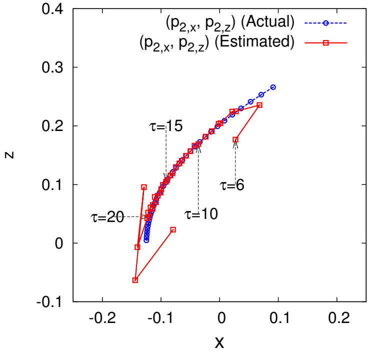


xz-**plane**

Estimation result for location of wave source 2



xy-**plane**



xz-**plane**

Further plans

- **Extend the method for other models of source**
- **Extend the result for limited aperture case**
- **Extend the method for other equations (e.g. diffusion equation, Maxwell equation, etc.)**