Homogenization in a Thin Layer with an Oscillating Interface and Highly Contrast Coefficients

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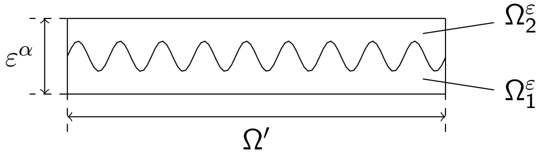
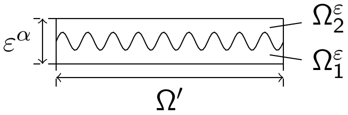
Problem setting: Geometry



- $\Omega' \subset \mathbb{R}^{n-1}$: a bounded domain
- $Y' := [0,1]^{n-1}$
- $g : \mathbb{R}^{n-1} \to \mathbb{R}$: smooth, periodic in Y', $0 < g < 1$. $\alpha > 0$.

$$\Omega^\varepsilon := \Omega' \times (0, \varepsilon^\alpha)$$
$$\Omega_1^\varepsilon := \{(x', x_n) \in \Omega^\varepsilon \,;\, 0 < x_n < \varepsilon^\alpha g(x'/\varepsilon)\}$$
$$\Omega_2^\varepsilon := \{(x', x_n) \in \Omega^\varepsilon \,;\, \varepsilon^\alpha g(x'/\varepsilon) < x_n < 1\}$$

Problem setting: formulation



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For $\beta > 0$,

$$a^\varepsilon(x) := \chi_{\Omega_1^\varepsilon}(x) + \varepsilon^\beta \chi_{\Omega_2^\varepsilon}(x).$$

For $\lambda > 0$, we consider the following problem:

$$\begin{cases} -\nabla \cdot (a^\varepsilon(x) \nabla u^\varepsilon(x)) + \lambda u^\varepsilon(x) = f^\varepsilon & \text{in } \Omega^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon. \end{cases} \tag{1}$$

Question: What is the limit of u^ε as $\varepsilon \to 0$?

Motivation

Homogenization in a thin layer

$$\Omega^\varepsilon := \Omega' \times (0, \varepsilon^\alpha)$$

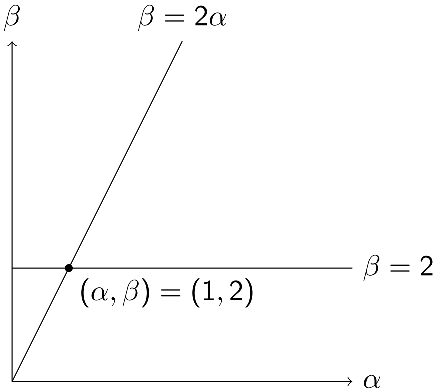
Double Porosity Problem

$$a^\varepsilon(x) := \chi_{\Omega_1^\varepsilon}(x) + \varepsilon^\beta \chi_{\Omega_2^\varepsilon}(x)$$

Our Problem

Main Point: Limit Problem depends on α and β.

Extremely rough summary of the main results



The behaviour of the limit of u^ε can be divided into nine regions on (α, β) space with two boundaries $\beta = 2\alpha$ and $\beta = 2$

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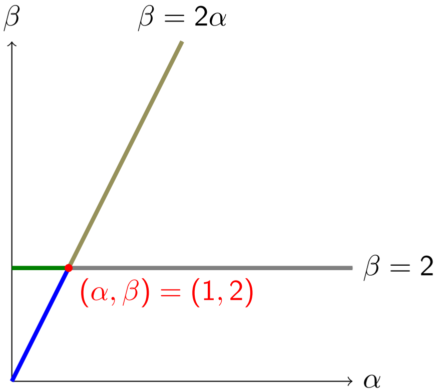

The behaviour of the limit of u^{ε} can be divided into nine regions on (α, β) space with two boundaries $\beta = 2\alpha$ and $\beta = 2$

In this talk, we focus on several cases which satisfies either $\beta = 2\alpha$ or $\beta = 2$.

Main result: Case: $\beta = 2$, $\alpha < 1$

Theorem

Let $\beta = 2$, $\alpha < 1$ and u^ε be a solution of the original problem (1). Then there exists a subsequence of (u^ε) (still denoted u^ε) such that $\frac{1}{\varepsilon^\alpha}\int_0^{\varepsilon^\alpha} u^\varepsilon(\cdot, x_n)\,\mathrm{d}x_n \rightharpoonup u$ weakly in $H_0^1(\Omega')$ as $\varepsilon \to 0$ and that u is a weak solution of the following equation:

$$
\begin{aligned}
-\nabla' \cdot (A^*\nabla' u) + \lambda\left(1 - \lambda\int_{Z_2} V(y)\,\mathrm{d}y_n\right) u = \\
\left(1 - \lambda\int_{Z_2} V(y)\,\mathrm{d}y_n\right) f, \quad (2)
\end{aligned}
$$

where V is a solution of

$$
\begin{cases}
-\Delta'_y V(y) + \lambda V(y) = 1 & \text{in } Z_1, \\
V(y) = 0 & \text{on } \{y_n = g(y')\}, \\
\partial_\nu V = 0 & \text{on } Y \times \{1\}, \\
y' \mapsto V(y', y_n) & \text{periodic in } Y \text{ for all } y_n,
\end{cases}
$$

Main result: Case $\beta = 2$ and $\alpha > 1$

Theorem

Let $\beta = 2$, $\alpha > 1$. Then the limit u is the weak solution of the following equation:

$$-\nabla' \cdot \left(a^* \nabla' u\right) + \lambda u = f,$$

where $a^* = \frac{1}{2} + \int_Y g(y')\,\mathrm{d}y'$.
In this case, non-local effect does NOT appear.